# LECTURE II

## Hamilton-Jacobi theory and Stäckel systems

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# Separability by Hamilton and Jacobi

- Consider Liouville integrable systems on symplectic manifold, given by $\{h_i, h_j\}_\pi = 0,\ i,j = 1, ..., n$ and related Hamiltonian systems
$$u_{t_i} = X_{h_i} = \pi dh_i, \qquad i = 1, ..., n, \quad u = (q,p)^T. \tag{2.1}$$
- Assume that $(q,p)$ are canonical (Darboux) coordinates. HJ method of solving (2.1) amounts to the linearization of (2.1) via a canonical transformation
$$(q,p) \longrightarrow (b,a), \qquad a_i = h_i, \qquad i = 1, ..., n. \tag{2.2}$$
- In order to find $b_i$ it is necessary to construct a generating function $W(q,a)$ of transformation (2.2)
$$b_i = \frac{\partial W}{\partial a_i}, \qquad p_i = \frac{\partial W}{\partial q_i}.$$
- The function $W(q,a)$ is an integral of associated HJ equations
$$h_i(q_1, ..., q_n, \frac{\partial W}{\partial q_1}, ..., \frac{\partial W}{\partial q_n}) = a_i, \qquad i = 1, ..., n.$$

# Separability by Hamilton and Jacobi

- In $(b, a)$ representation dynamical systems (2.1) are trivial

$$(a_j)_{t_i} = 0, \qquad (b_j)_{t_i} = \delta_{ij}, \qquad i, j = 1, \ldots, n,$$

and

$$b_j(q, a) = \frac{\partial W}{\partial q_j} = t_j + c_j, \qquad j = 1, \ldots, n. \tag{2.3}$$

- Eqs.(2.3) provide implicit solutions of (2.1) in oryginal coordinates. Solving it for $q$ we reconstruct in explicit form trajectories

$$q_i = q_i(t_1, \ldots, t_n, a_1, \ldots, a_n, c_1, \ldots, c_n).$$

Inverse Jacobi problem.

- Where is the hook? How to overcome it?

# Separability by Hamilton and Jacobi

- Find a distinguished Darboux coordinates $(\lambda, \mu)$, for which there exist $n$ relations (separation relations)

$$\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \qquad i = 1, \ldots, n, \tag{2.4}$$

$$a_i \in \mathbb{R}, \qquad \left|\frac{\partial \varphi_i}{\partial a_j}\right| \neq 0,$$

- which can be solved with respect to $a_i$:

$$a_i = h_i(\lambda, \mu),$$

  reconstructing our Hamiltonians in new coordinates.
- In fact one can prove that any set of algebraic equations (2.4) defines a Lagrangian foliation of symplectic manifold.

# Separability by Hamilton and Jacobi

- We are looking for a generating function $W(\lambda, a)$ of transformation $(\lambda, \mu) \longrightarrow (b, a)$ in the form

$$W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda_i, a_1, ..., a_n),$$

where functions $W_i$ are solutions of a system of $n$ decoupled ODEs obtained from separation relations under substitution $\mu_i = \frac{dW_i}{d\lambda_i}$

$$\varphi_i(\lambda_i, \mu_i = \frac{dW_i}{d\lambda_i}, a_1, ..., a_n) = 0, \qquad i = 1, ..., n.$$

- Such an additively separable solution $W(\lambda, a)$ is simultanously the solution of all Hamilton Jacobi equations. $(\lambda, \mu)$ coordinates are called separation coordinates.

# Stäckel systems and their classyfication

- Consider separation relations affine in Hamiltonians $h_i$:

$$\sum_{k=1}^{n} S_i^k(\lambda_i, \mu_i) h_k = \psi_i(\lambda_i, \mu_i), \qquad i = 1, ..., n, \tag{2.5}$$

  called generalized Stäckel separation relations. $S = \left(S_i^k\right)$ - generalized Stäckel matrix, $\psi = (\psi_i)$ - generalized Stäckel vector.
- If $S_i^k(\lambda_i, \mu_i) = S^k(\lambda_i, \mu_i)$ and $\psi_i(\lambda_i, \mu_i) = \psi(\lambda_i, \mu_i)$, then (2.5) can be represented by $n$ copies of the curve

$$\sum_{k=1}^{n} S^k(\lambda, \mu) h_k = \psi(\lambda, \mu) \tag{2.6}$$

  in $(\lambda, \mu)$ plane, called separation curve.
- In particular, if (2.6) is nonsingular, compact Riemann Surface $\Gamma$, then one can find genus of this curve, basic holomorfic differentials and solve the inverse Jacobi problem in the language of Riemann theta functions.

# Stäckel systems and their classyfication

- Let us write separation relations in the form

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i, \mu_i) \left[\gamma^{(k)}(\lambda_i) + h^{(k)}(\lambda_i, n_k)\right] = \chi_i(\lambda_i, \mu_i), \qquad i = 1, \ldots, n,$$

$$h^{(k)}(\lambda, n_k) = \sum_{j=1}^{n_k} h_j^{(k)} \lambda^{n_k - j}, \qquad n_1 + \ldots + n_m = n, \qquad \varphi_i^m(\lambda_i, \mu_i) = 1. \tag{2.7}$$

- They split onto bare part

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i, \mu_i) \left[E^{(k)}(\lambda_i, n_k)\right] = \chi_i(\lambda_i, \mu_i), \qquad i = 1, \ldots, n,$$

and generalized separable "potentials"

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i, \mu_i) \left[\gamma^{(k)}(\lambda_i)\delta_s^k + V^{(k,s)}(\lambda_i, n_k)\right] = 0, \qquad i = 1, \ldots, n.$$

# Stäckel systems and their classyfication

- Basic potentials: $\gamma^{(s)}(\lambda) = \lambda^{r_s}$, $r_s \in \mathbb{Z}$, $s = 1, \ldots, m$. Then

$$h_i^{(k)} = E_i^{(k)} + \sum_{s=1}^{m} V_i^{(k,s,r_s)}.$$

- There are $m$ hierarchies of basic potentials.
- Particular class - fixed Stäckel matrix $S$ and vector $\chi$. $S$ is determined uniquely by $m$ vectors $\varphi^k = (\varphi_1^k, \ldots, \varphi_n^k)$, $k = 1, \ldots, m$ and the partition of $n : (n_1, \ldots, n_m)$.
- Classical Stäckel systems: one particle dynamics on pseudo-Riemann space

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i) \left[\gamma^{(k)}(\lambda_i) + h^{(k)}(\lambda_i, n_k)\right] = \frac{1}{2} f_i(\lambda_i)\mu_i^2, \qquad i = 1, \ldots, n.$$

# Stäckel systems and their classyfication

- Benenti class: $m = 1$

$$\gamma(\lambda_i) + \underbrace{h_1\lambda_i^{n-1} + \dots + h_n}_{h(\lambda,n)} = \frac{1}{2} f_i(\lambda_i)\mu_i^2, \qquad i = 1, \dots, n$$

contains majority of known classical separable systems in flat or constant curvature Riemann spaces, with all constants of motion quadratic in momenta.

- Other classes with $m = 1$:

$$\gamma(\lambda_i) + \underbrace{h_1\lambda_i^{n-1} + \dots + h_n}_{h(\lambda,n)} = \left\{ \begin{array}{c} f_i(\lambda_i)\mu_i^3 \\ \exp(a\mu_i) + \exp(-b\mu_i) \end{array} \right.$$

Periodic Toda, KdV dressing chain, relativistic $n$ - body problem, ...

- The case with $m = 2$ :

$$\mu_i \left[\gamma^{(1)}(\lambda_i) + h^{(1)}(\lambda_i, n_1)\right] + \left[\gamma^{(2)}(\lambda_i) + h^{(2)}(\lambda_i, n_2)\right] = f_i(\lambda_i)\mu_i^3,$$

stationary flows of Bussinesq: $n_1 = 1,\ n_2 = n - 1$, dynamical system on loop algebra $sl(3)$ : $\quad n_1 = 2,\ n_2 = 4.$

# Classical Stäckel systems

- $(Q,g)-$ pseudo Riemann space and dynamical system

$$q^i_{tt}+\Gamma^i_{jk}q^j_t q^k_t = G^{ik}\partial_k V(q), \quad i=1,...,n \tag{2.8}$$

$$\Updownarrow$$

$$q^i_t=\frac{\partial h}{\partial p_i}, \quad (p_i)_t=-\frac{\partial h}{\partial q^i}, \qquad h(q,p)=\frac{1}{2}\sum_{i,j}G^{ij}p_ip_j+V(q).$$

- Assume that (2.8) is Liouville integrable with all constants of motion quadratic in momenta:

$$h_r(q,p)=\frac{1}{2}\sum_{i,j}(K_rG)^{ij}\,p_ip_j+V_r(q), \quad r=1,...,n,$$

where $K_1=I,\ h_1=h,\ K_r-$ Killing tensors.

- The transformation to separation coordinates $(\lambda,\mu)$ is a point transformation generated by

# Classical Stäckel systems

- $$F(p,\lambda)=\sum_{i=1}^{n}\theta_i(\lambda)p_i \Longrightarrow \quad q^i=\frac{\partial F}{\partial p_i}=\theta_i(\lambda), \quad \mu_i=\frac{\partial F}{\partial \lambda_i}=p_i\frac{\partial \theta_i(\lambda)}{\partial \lambda_i}.$$
- The explicit form of Hamiltonians in separation coordinates depends on the form of separation relations.
- **Example.** Benenti class:

$$G^{ij}=\frac{f_i(\lambda_i)}{\Delta_i}\delta^{ij}, \quad (K_r)^i_j=-\frac{\partial \rho_r}{\partial \lambda_i}\delta^i_j, \quad V_r=\sum_i \frac{\partial \rho_r}{\partial \lambda_i}\frac{\gamma_i(\lambda_i)}{\Delta_i},$$

- where

$$\Delta_i=\prod_{k\neq i}(\lambda_i-\lambda_k)$$

and $\rho_r$ are Viete polynomials:
$\rho_1=-(\lambda_1+...+\lambda_n), \ ..., \rho_n=(-1)^n\lambda_1...\lambda_n.$

# Classical Stäckel systems

- Linearization in $(b, a)-$ coordinates.
- Consider subclasses

$$\lambda_i^k + h_1\lambda_i^{\gamma_1} + h_2\lambda_i^{\gamma_2} + \dots + h_n = \frac{1}{2}f_i(\lambda_i)\mu_i^2, \qquad i = 1, \dots, n \qquad (2.9)$$

- and generating function

$$W(a,\lambda) = \sum_i W_i(\lambda_i, a) \to \ \mu_i = \frac{dW_i}{d\lambda_i}, \quad b_i = \frac{\partial W}{\partial a_i}.$$

- Hence, from (2.9)

$$\frac{1}{2}f_i(\lambda_i)\left(\frac{dW_i}{d\lambda_i}\right)^2 = \lambda_i^k + \sum_{r=1}^{n} a_r\lambda_i^{\gamma_r} \equiv P(\lambda_i, a)$$

$$\Downarrow$$

$$b_i = \frac{\partial W}{\partial a_i} = \sum_{j=1}^{n}\int^{\lambda_j} \frac{\xi^{\gamma_i}\, d\xi}{\sqrt{R_j(\xi, a)}} = t_i + c_i, \qquad i = 1, \dots, n$$

# Classical Stäckel systems

- where $R_j(\xi, a) = 2f_j(\xi)P(\xi, a)$, or in Abel-Jacobi differential form

$$\sum_{j=1}^{n} \frac{\lambda_j^{\gamma_i} d\lambda_j}{\sqrt{R_j(\lambda_j, a)}} = dt_i, \quad i = 1, \ldots, n.$$

- **Examples.**
- Natural Hamiltonians of one-degree of freedom

$$h = \frac{1}{2m} p^2 + V(x). \tag{2.10}$$

$(x, p)$ are separation coordinates and (2.10) itselve represents separation relation.

- Take harmonic oscillator with $V(x) = \frac{1}{2}\alpha x^2$. Then

$$\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + \frac{1}{2}\alpha x^2 = a \quad \Longrightarrow \quad W = \int \sqrt{2m(a - \frac{1}{2}\alpha x^2)} dx$$

# Classical Stäckel systems

- $$\Longrightarrow \quad t+c=\frac{dW}{da}=\int\frac{mdx}{\sqrt{2m(a-\frac{1}{2}\alpha x^2)}}=\sqrt{\frac{m}{\alpha}}\arcsin\sqrt{\frac{\alpha}{2a}}x$$

  $$\Longrightarrow \quad x=A\sin(\omega t+\varphi), \quad A=\sqrt{\frac{2a}{\alpha}}, \;\; \omega=\sqrt{\frac{\alpha}{m}}, \;\; \varphi=\omega c.$$

- For potential $V(x)=\frac{1}{2}\alpha x^2-\frac{1}{4}\beta x^4$ the similar calculation gives

  $$x=Asn(\omega t+\varphi,k)$$

  sinus eliptic function of Jacobi, where $(A,\omega,k)$ are expressed by $(\alpha,\beta,a)$ through

  $$m\beta A^2=2k^2\omega^2, \quad m\alpha=\omega^2(1+k^2), \quad 2a=A^2\omega^2, \;\; \varphi=\omega c.$$

# Classical Stäckel systems

- Henon-Heiles system (two degrees of freedom)

$$h \equiv h_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1^3 + \frac{1}{2}q_1 q_2^2,$$

$$h_2 = \frac{1}{2}q_2 p_1 p_2 - \frac{1}{2}q_1 p_2^2 + \frac{1}{4}q_1^2 q_2^2 + \frac{1}{16}q_2^2.$$

- Transformation to separation coordinates

$$q_1 = \lambda_1 + \lambda_2, \qquad p_1 = \frac{\lambda_1 \mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2 \mu_2}{\lambda_2 - \lambda_1},$$

$$q_2 = \sqrt{-4\lambda_1\lambda_2}, \qquad p_1 = \sqrt{-\lambda_1\lambda_2}\left(\frac{\mu_1}{\lambda_1 - \lambda_2} + \frac{\mu_2}{\lambda_2 - \lambda_1}\right).$$

# Classical Stäckel systems

- Separation relations

$$h_1\lambda_1 + h_2 = \frac{1}{2}\lambda_1\mu_1^2 + \lambda_1^4$$
$$h_1\lambda_2 + h_2 = \frac{1}{2}\lambda_2\mu_2^2 + \lambda_2^4.$$

- Implicit solution in Abel-Jacobi form

$$dt_1 = \frac{\lambda_1 d\lambda_1}{\sqrt{R(\lambda_1, a)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{R(\lambda_2, a)}}$$
$$dt_2 = \frac{d\lambda_1}{\sqrt{R(\lambda_1, a)}} + \frac{d\lambda_2}{\sqrt{R(\lambda_2, a)}},$$

where $R(\lambda, a) = a_2 + a_1\lambda - \lambda^4$.